

I got rhythm:
Mathematics and Music

Douglas Butler - violin
Mick Blaylock - keyboard

Description for the Conference website (600 characters)

- From vibrating strings to jazz piano via the 'Well-tempered clavier'.
- Starting with a vibrating string, we consider some of the underlying mathematics of the string and its harmonics. We explore the development of the notes of the keyboard, the intrigue and mystery of the 'Well tempered Clavier'.
- We will also explore how mathematics relates to rhythm and music-making with reference to popular songs and development of musical ideas and harmony. These ideas will be illustrated with music from blues, ragtime, musicals, rock and roll and jazz performed by both Douglas (violin) and Mick (piano)

Musical Numbers Quiz

- 440
- 12
- 251
- 88
- 1645
- 1812
- $\frac{3}{4}$
- 2, 3, 5, 8, 13, ...

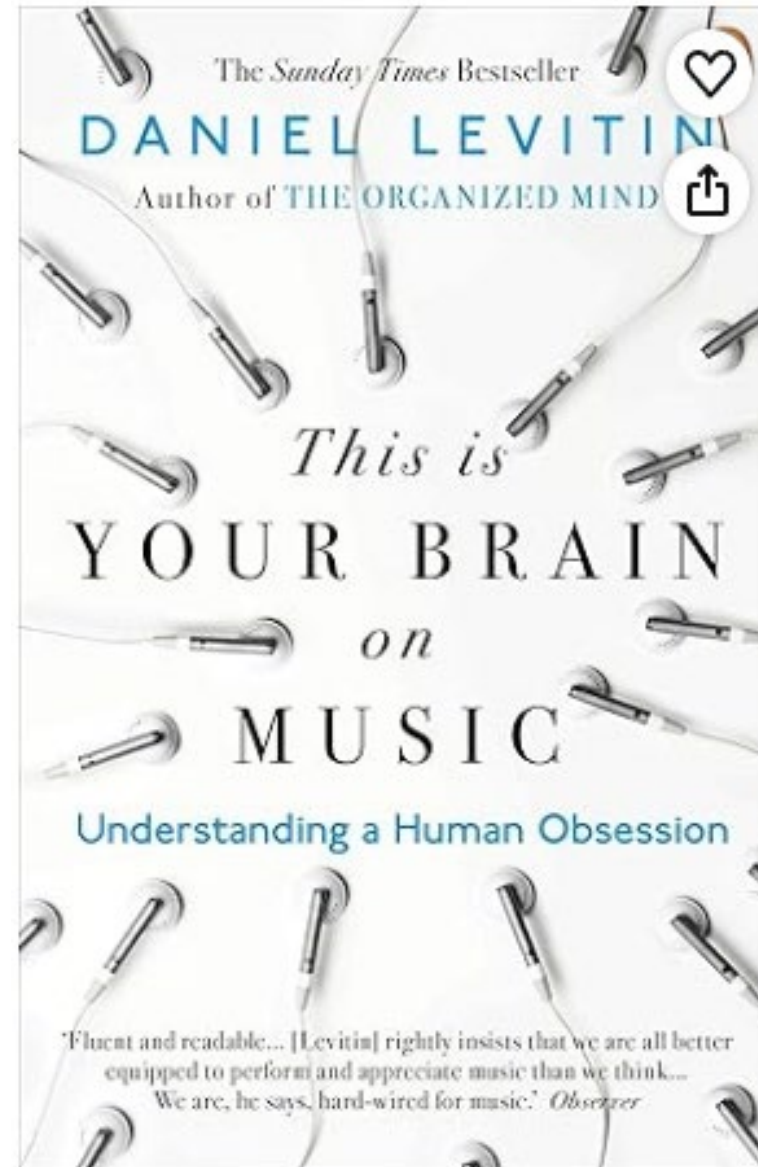


Embrace
able You
– George
Gershwin

What is Music?

Essential components:

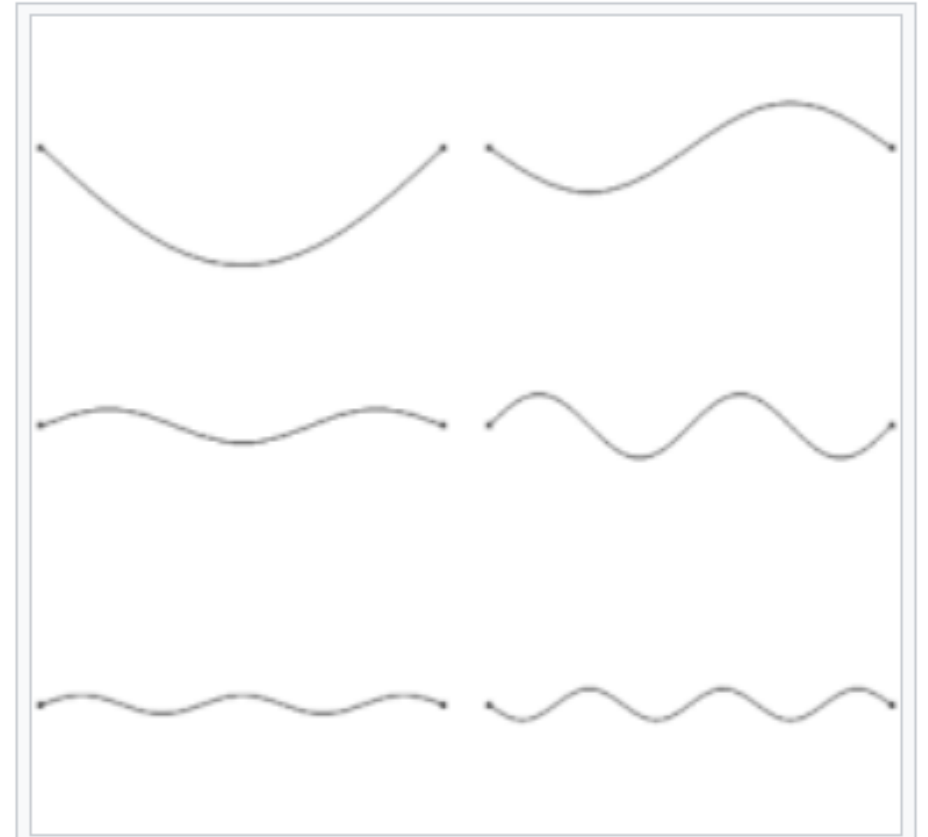
- Pitch
- Timbre
- Rhythm



Pitch: harmonics and frequencies ...



- Vibrating string
- https://en.wikipedia.org/wiki/String_vibration



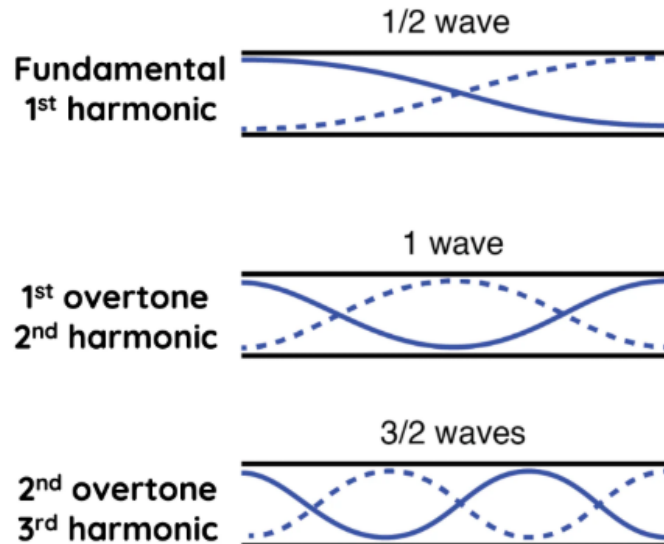
Standing waves in a pipe 1

Pipe with two open ends

The lowest frequency required to produce a standing wave is the **fundamental frequency**, f_0 , also called the **first harmonic**.

Frequencies required to form the standing wave are integral multiples of the fundamental frequency:

$$f = nf_0 \text{ where } n = 1, 2, 3...$$

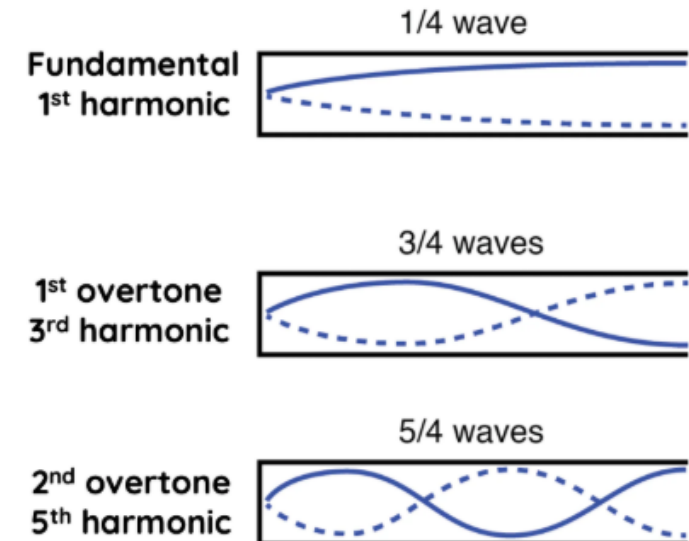


Pipe with one closed end

Frequencies required to form a standing wave in this system are given by

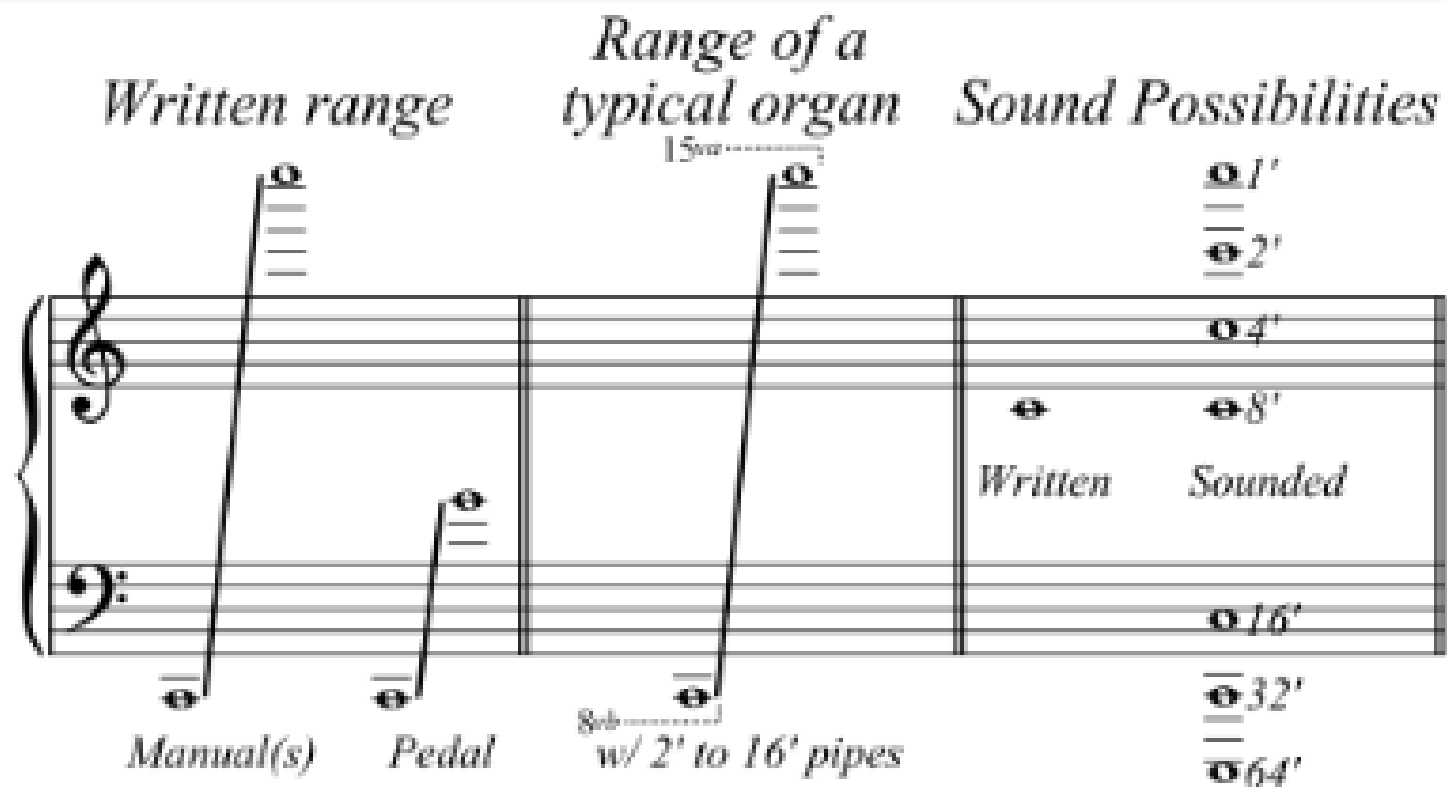
$$f = nf_0 \text{ where } n = 1, 3, 5...$$

Note that n must be an odd integer when one end of the pipe is closed.

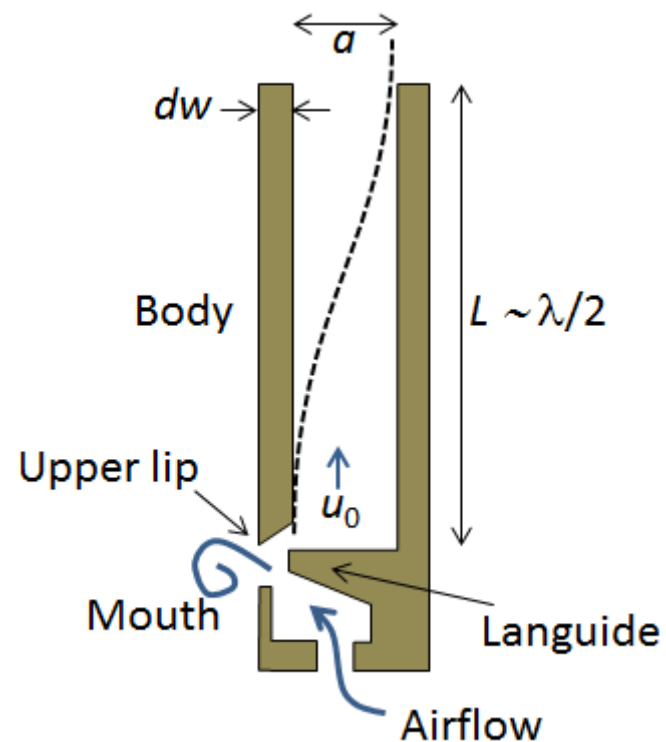


The Pipe Organ

Playing range



Standing waves in a pipe 2



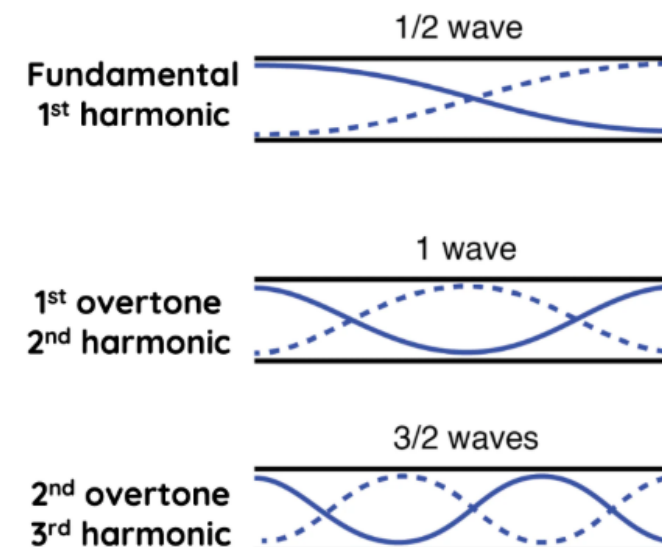
- Brass instruments are modelled as **open pipes**, specifically functioning as acoustic resonators.
- Even though a player's lips close one end (the mouthpiece), the buzzing creates a physical displacement antinode at the lips.
- This behaves exactly like an **open end** in acoustic theory, allowing the instrument to produce a full harmonic series.

Pipe with two open ends

The lowest frequency required to produce a standing wave is the **fundamental frequency**, f_0 , also called the **first harmonic**.

Frequencies required to form the standing wave are integral multiples of the fundamental frequency:

$$f = n f_0 \text{ where } n = 1, 2, 3 \dots$$



Frequencies for the Pythagorean scale

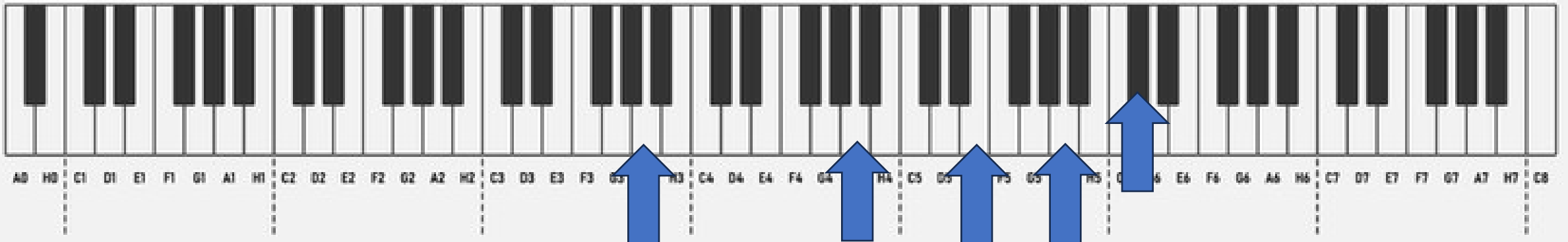
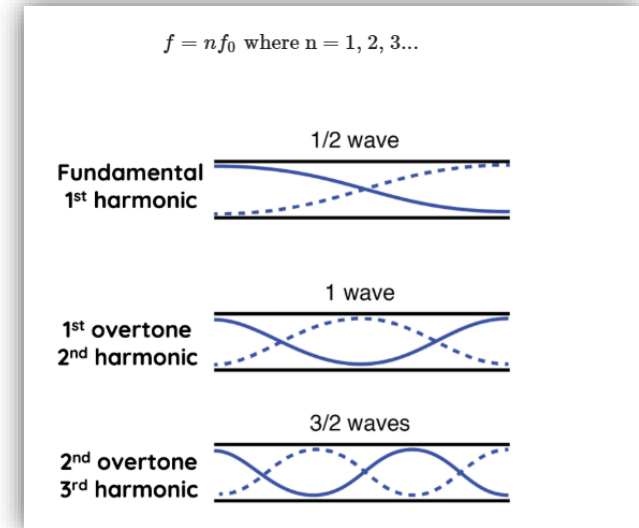
Taking 1st harmonic Concert A at 440 Hz

2nd harmonic is A, one octave higher at $2 \times 440 = 880$ Hz

3rd harmonic is note E in second octave
 $3 \times 440 = 1320$ Hz

4th harmonic is A, two octaves higher at $4 \times 440 = 1760$ Hz

5th harmonic is C#, in third octave at $5 \times 440 = 2200$ Hz



Frequencies within one octave for the Pythagorean scale

Concert A 440 Hz

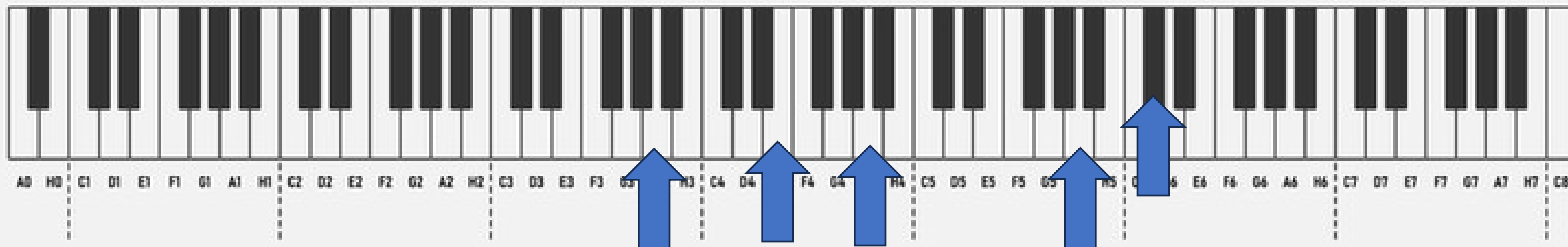
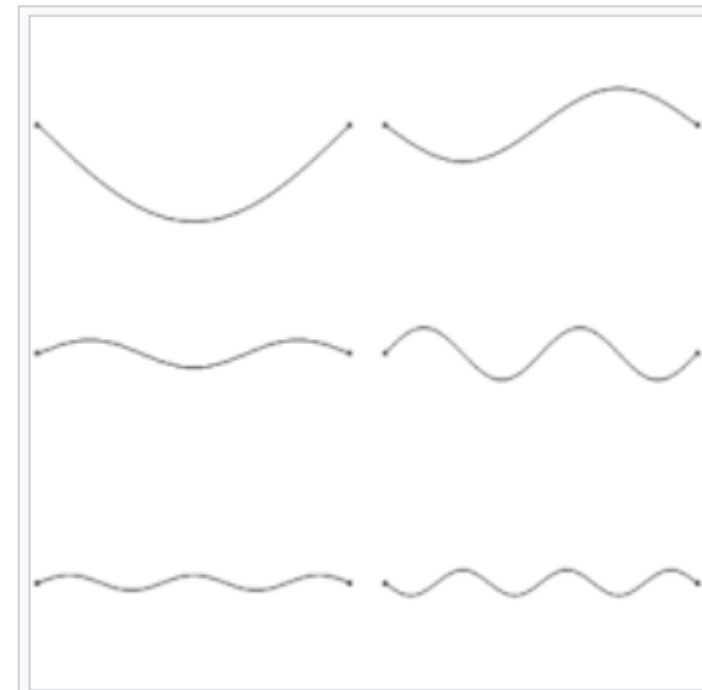
Note E $1320 \div 2 = 660\text{Hz}$

Note A one octave higher at 880 Hz

3rd harmonic is note E in second octave $3 \times 440 = 1320\text{Hz}$

4th harmonic is A, two octaves higher at $4 \times 440 = 1760\text{ Hz}$

5th harmonic is C#, in third octave at $5 \times 440 = 2200\text{ Hz}$



Frequencies within one octave for the Pythagorean scale

Concert A 440 Hz

Note C# $2200 \div 4 = 550\text{Hz}$

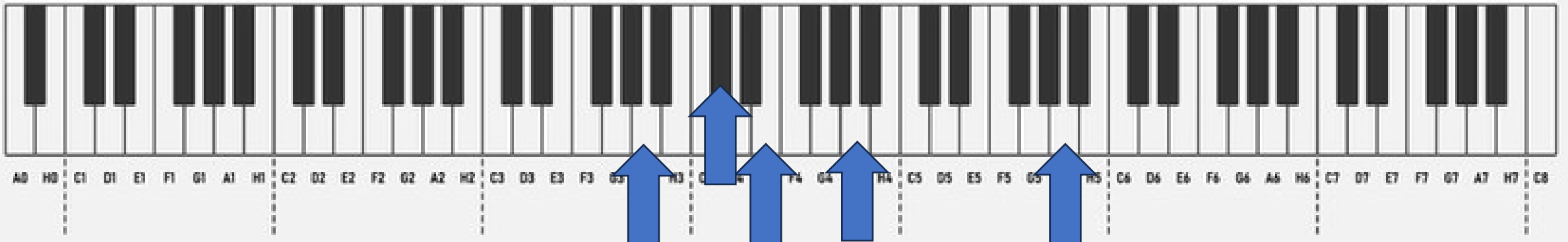
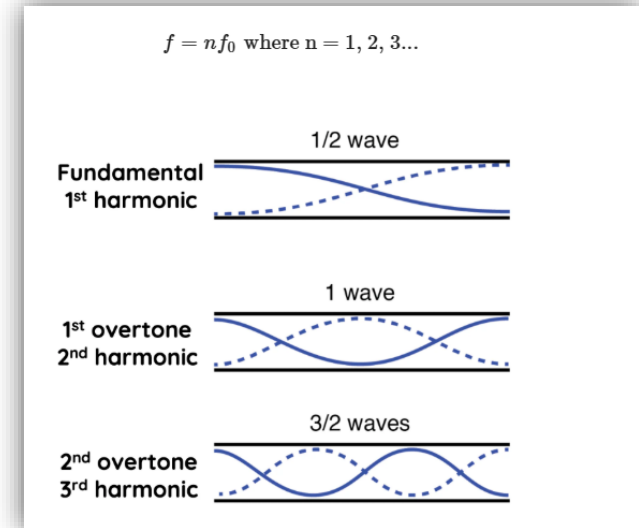
Note E $1320 \div 2 = 660\text{Hz}$

Note A one octave higher at 880 Hz

3rd harmonic is note E in second octave $3 \times 440 = 1320\text{Hz}$

4th harmonic is A, two octaves higher at $4 \times 440 = 1760\text{ Hz}$

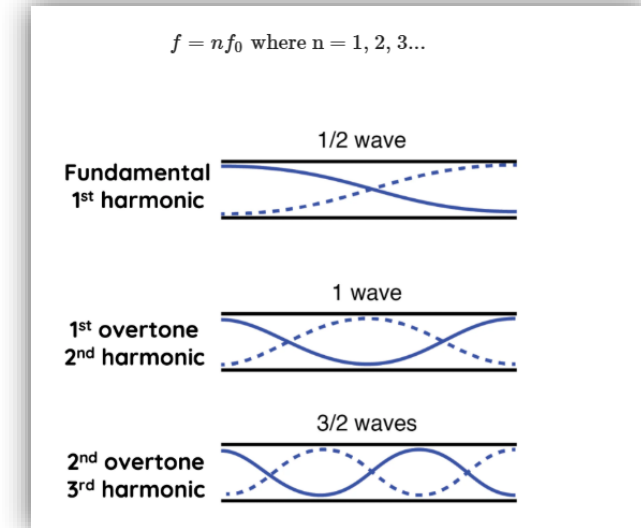
5th harmonic is C#, in third octave at $5 \times 440 = 2200\text{ Hz}$



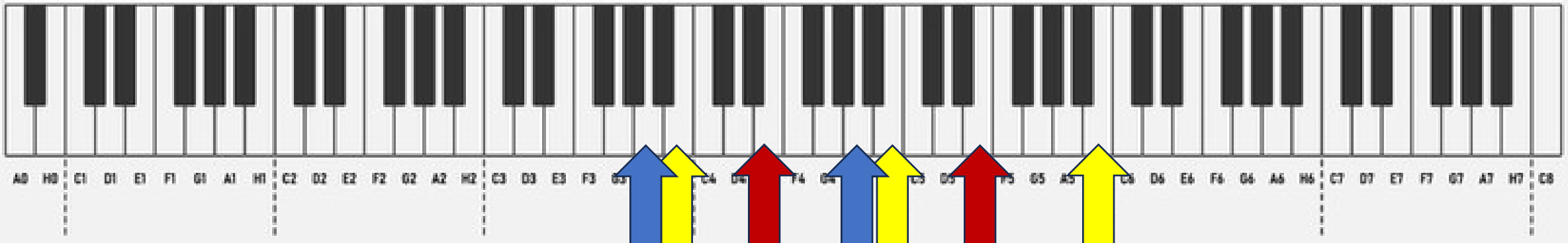
Iterative process using “fifths” to generate Frequencies

Multiply by 3 to find the frequency one fifth above
Divide by 2 (once or twice) to fit into the same octave

Concert A		440 Hz
E	$1320 \div 2 =$	660 Hz
A one octave higher at		880 Hz
B in third octave	$660 * 3 =$	1980 Hz
B in second octave	$1980 \div 2 =$	990 Hz
B in first octave	$990 \div 2 =$	495 Hz



Complete table of frequencies in handout for the notes in the scale of A major

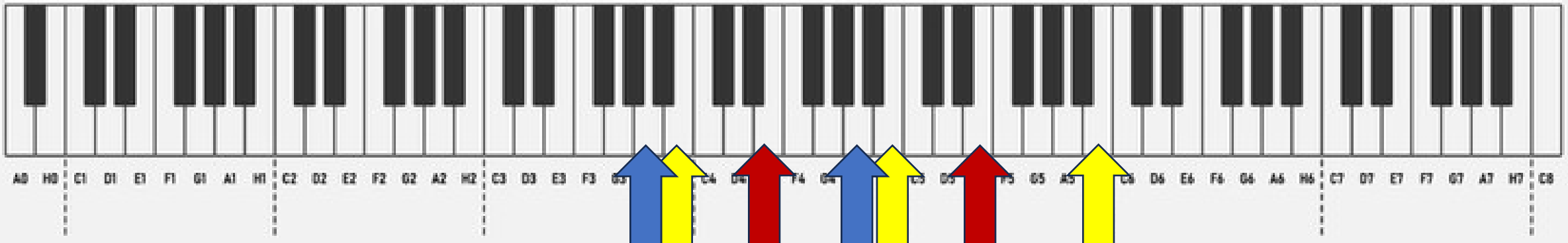
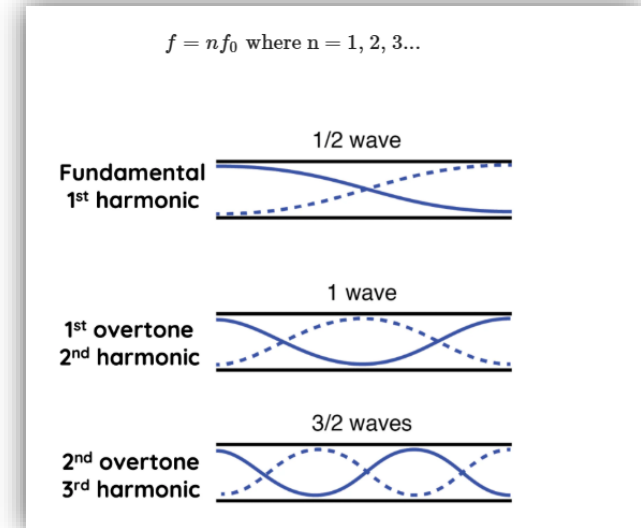


Using “fifths” to generate Frequencies

Complete the table

A	440 Hz
B	495 Hz
C#	
D	
E	660 Hz
F#	
G#	

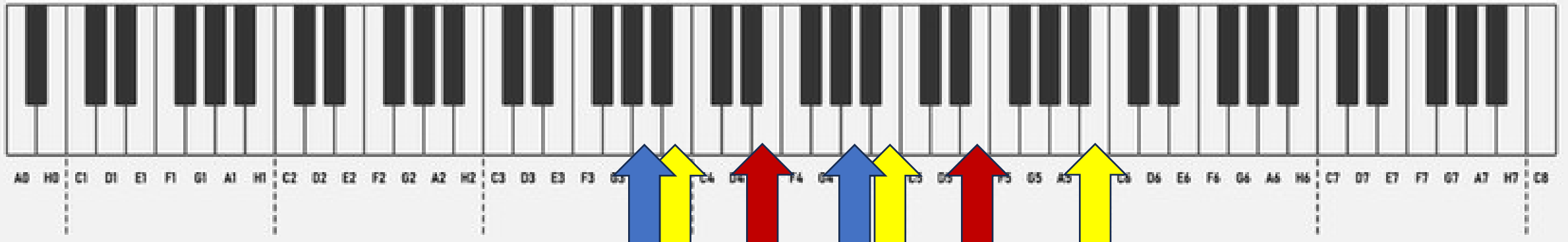
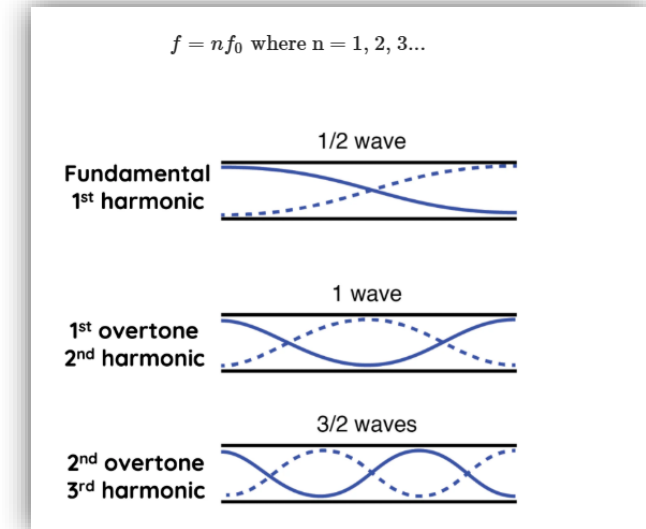
A one octave higher at 880 Hz



Using “fifths” to generate Frequencies for the Pythagorean scale

Completed table

A	440
B	495
C#	556.875
D	586.6666667
E	660
F#	742.5
G#	835.3125
A	880



Ratio of frequencies for the Pythagorean scale

A major scale	Freq (Hz)	Ratio to fundamental
A	440	1
B	495	1.125
C#	556.875	1.265625
D	586.6667	1.333333
E	660	1.5
F#	742.5	1.6875
G#	835.3125	1.898438
A	880	2

A major scale	Freq (Hz)	Ratio of adjacent notes
A	440	
B	495	1.125
C#	556.875	1.125
D	586.6667	1.053498
E	660	1.125
F#	742.5	1.125
G#	835.3125	1.125
A	880	1.053498

Tone ratio
1.125

Semitone ratio
1.053498

Ratio of frequencies for the Pythagorean scale: problems!

Tone ratio 1.125

Semitone ratio 1.053498

$$1.053498^2 = 1.1099 \neq 1.125$$

One octave = 7 tones $1.125^6 = 2.02728653$ close to 2

One octave = 13 semitones $1.053498^{12} = 1.868978479$ not
close to 2

Ratio of frequencies for the equal tempered scale

Let r be the ratio for a semitone. Then

$$r^{12} = 2$$

$$r = \sqrt[12]{2}$$

$$r = 1.059463$$

Close to Pythagorean semitone ratio of 1.053498

Concert A at 440 Hz

Frequencies for the equal tempered A scale

Using $r = 1.059463$

A	440
A#/Bflat	466.1637615
B	493.8833013
C	523.2511306
C#	554.365262
D	587.3295358
D#	622.2539674
E	659.2551138
F	698.4564629
F#	739.9888454
G	783.990872
G#	830.6093952
A	880

	Equal Temper	From fifths
A	440	440
B	493.8833	495
C#	554.3653	556.875
D	587.3295	586.6667
E	659.2551	660
F#	739.9888	742.5
G#	830.6094	835.3125
A	880	880

Tone generator

<https://onlinesound.net/tone-generator>

Waveform:

☒ Sine (Default)



☐ Square



☐ Triangle



☐ Sawtooth

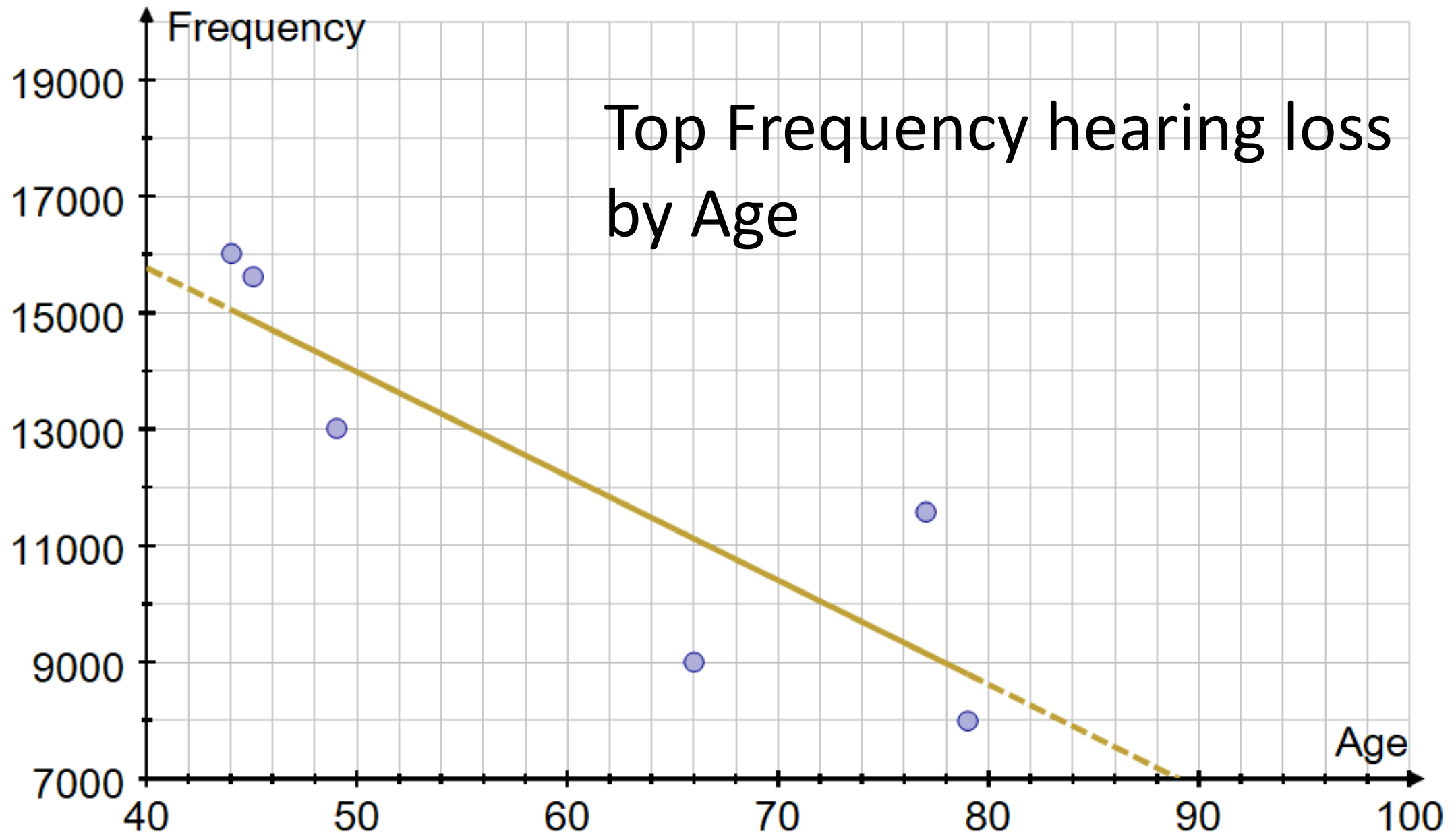


Top Frequency hearing loss by Age



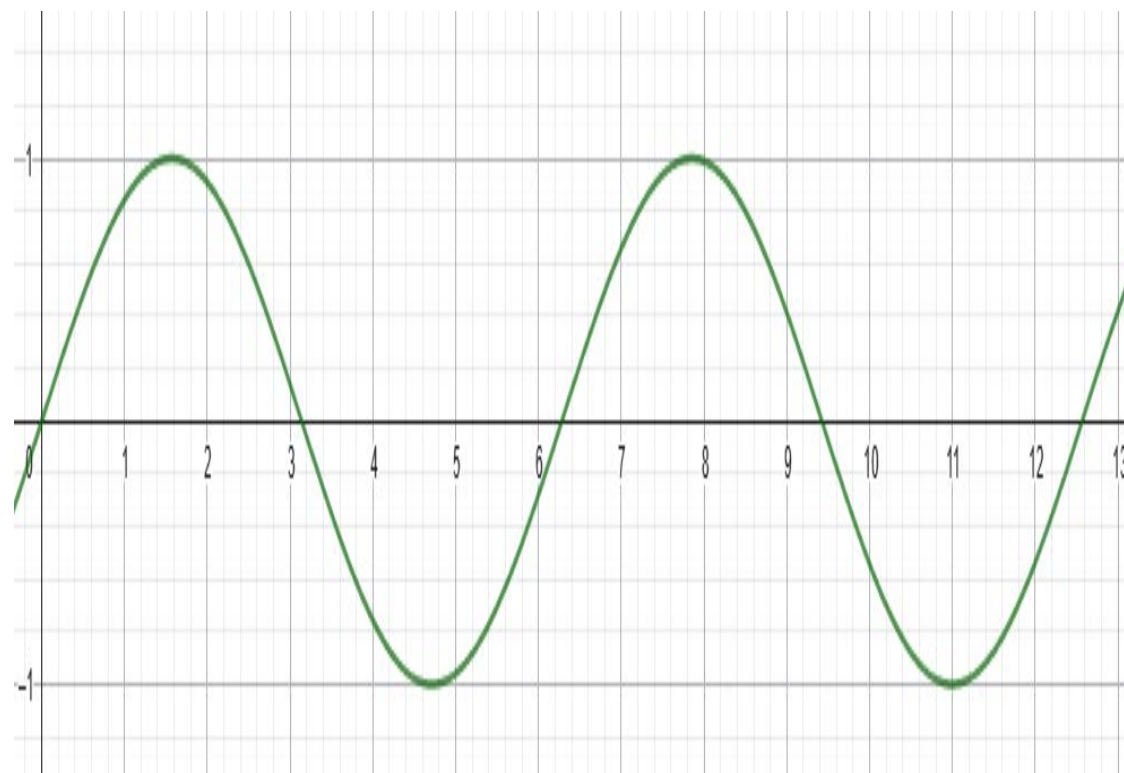
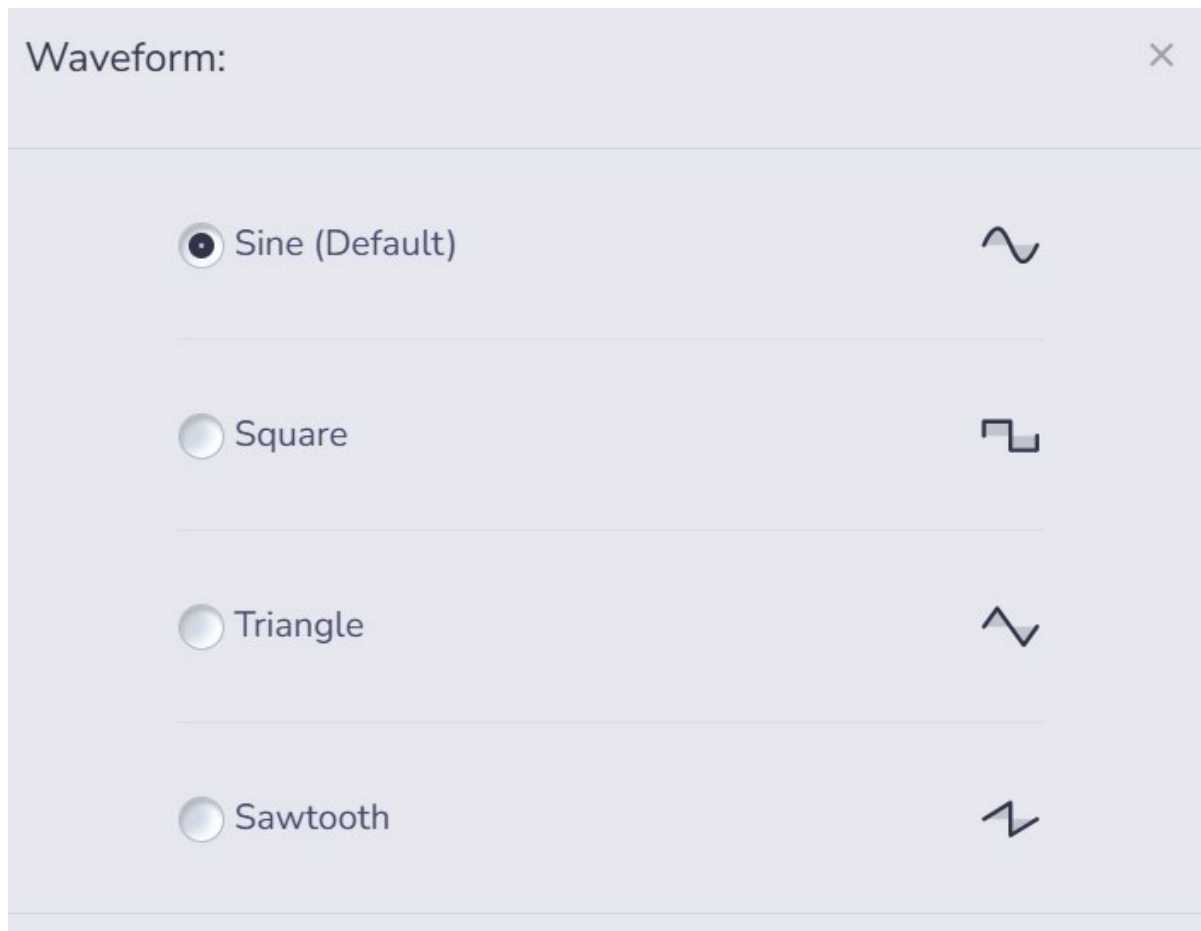
Tone Generator Online Tool 1-20000 Hz

Best classical generator of tone sounds of a given frequency with wave visualization. Generate clear sine wave sound signal. Full frequency range from **sub bass** to **high treble** (1-20000 Hz). Free web-app **no install**. Take and use!



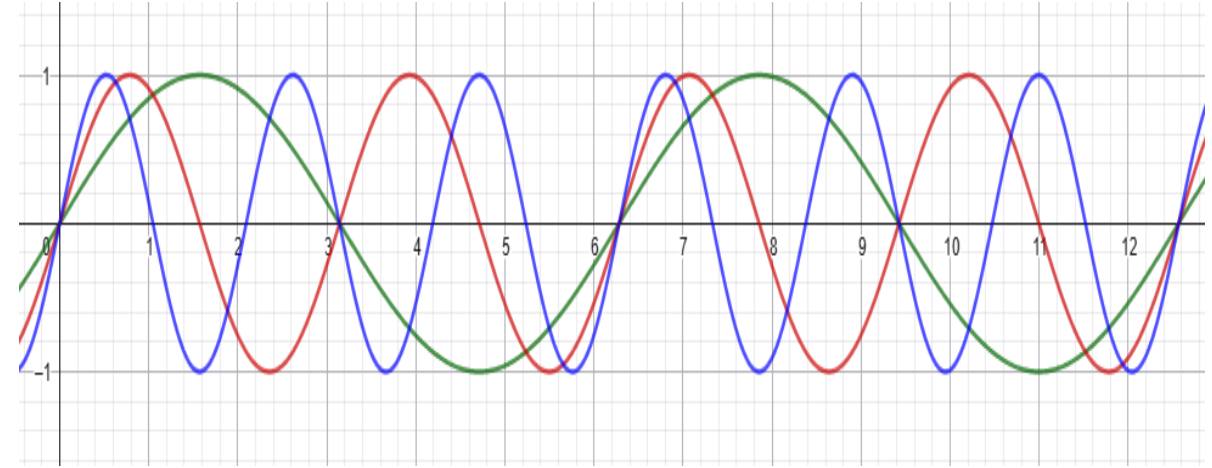
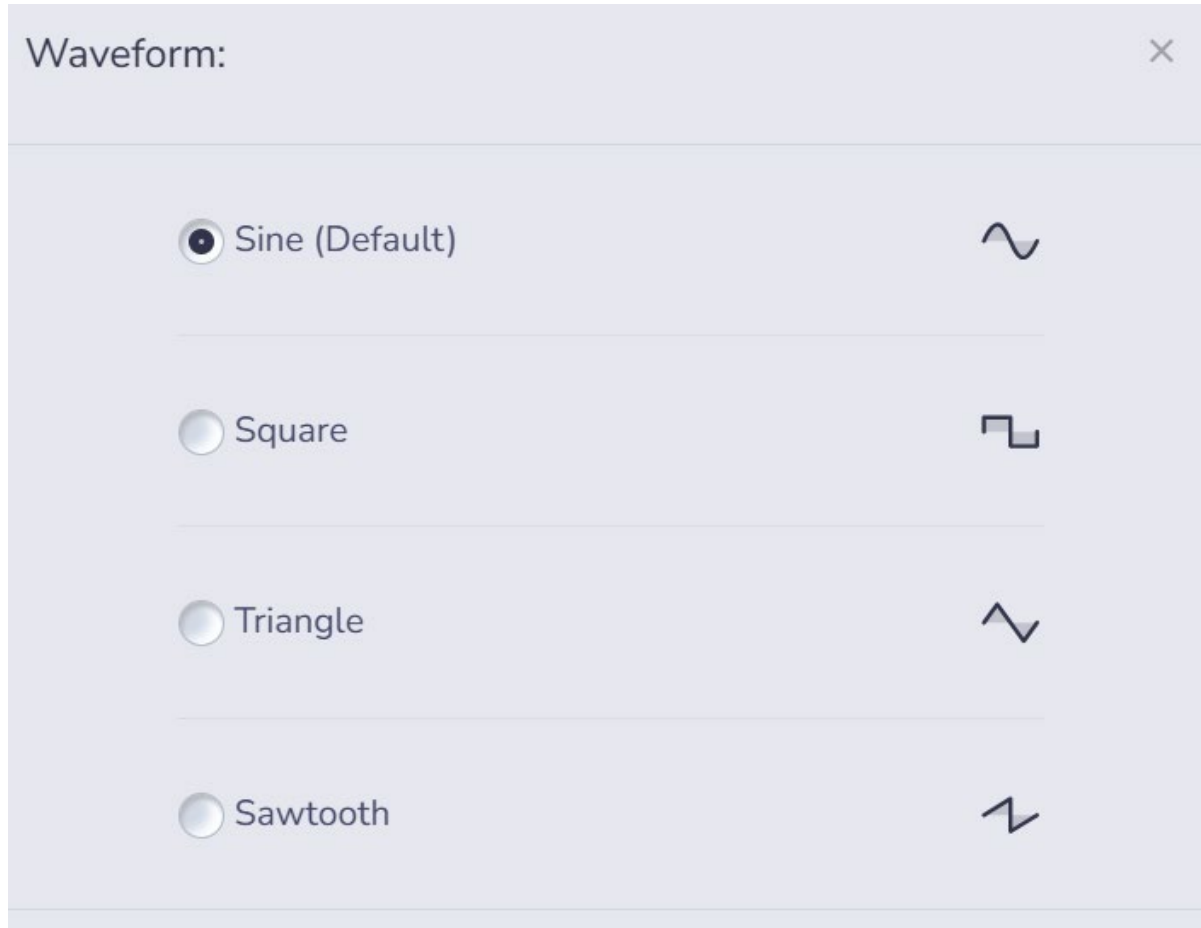
Timbre

$$\underline{y=\sin(x)}$$



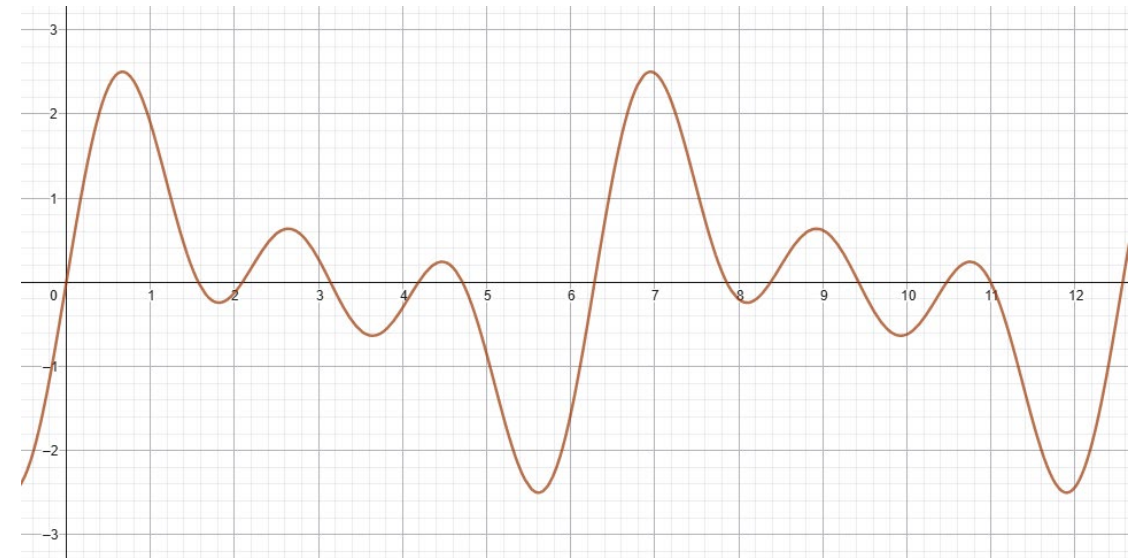
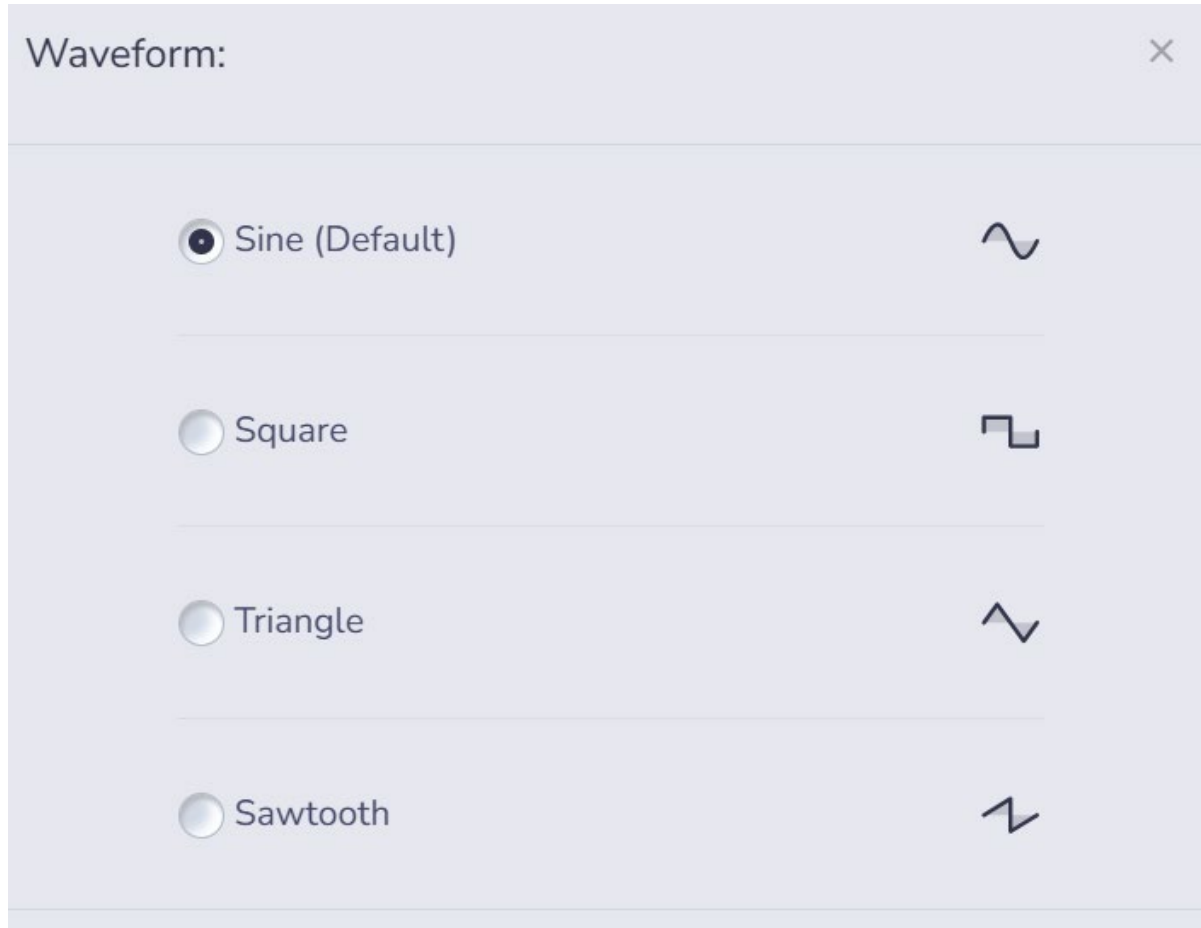
Timbre

Graphs of $\sin(x)$, $\sin(2x)$, $\sin(3x)$



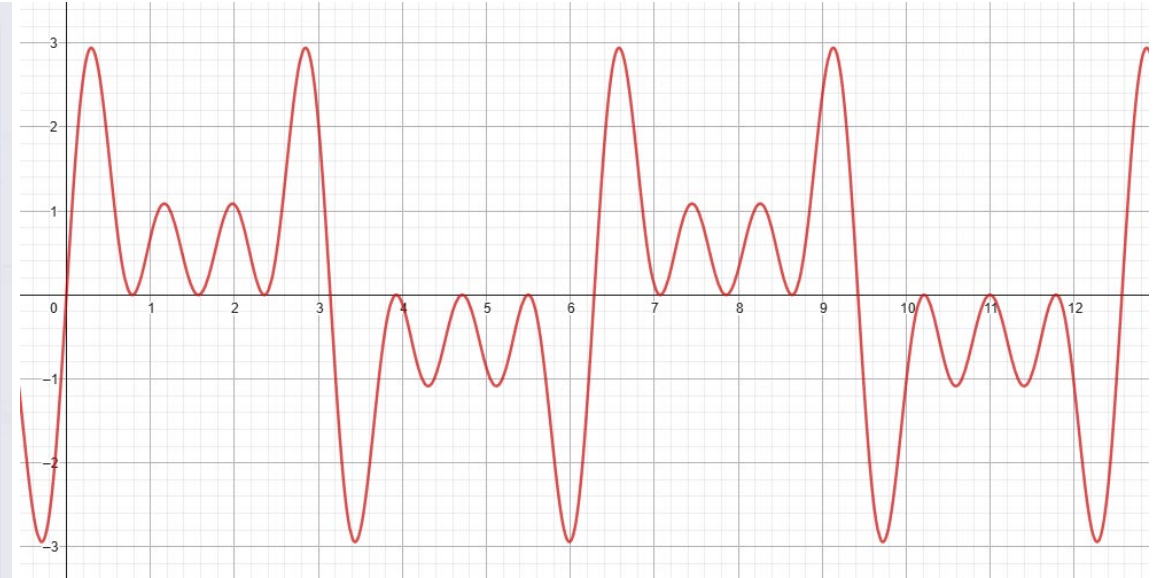
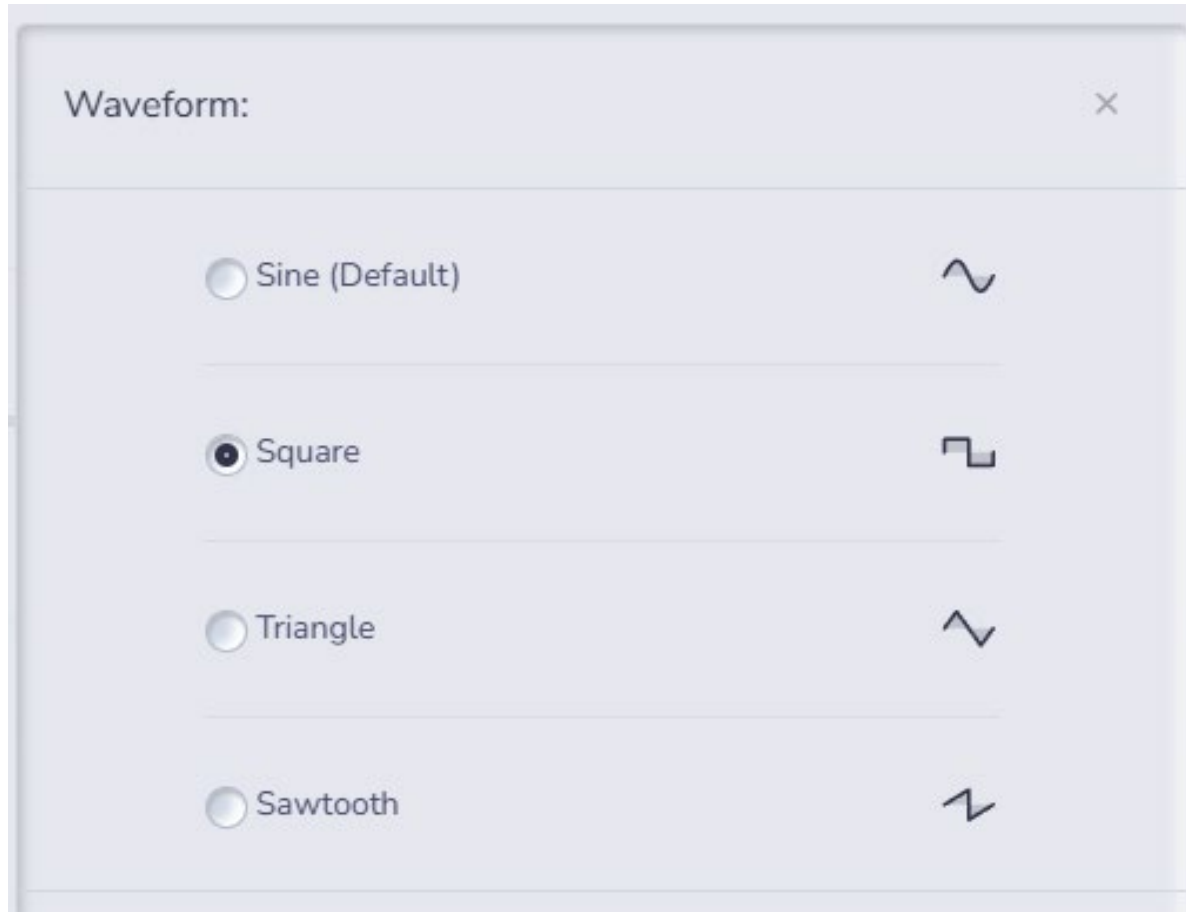
	$f(x) = \sin(x)$
	$g(x) = \sin(2x)$
	$h(x) = \sin(3x)$

Timbre combining sine graphs



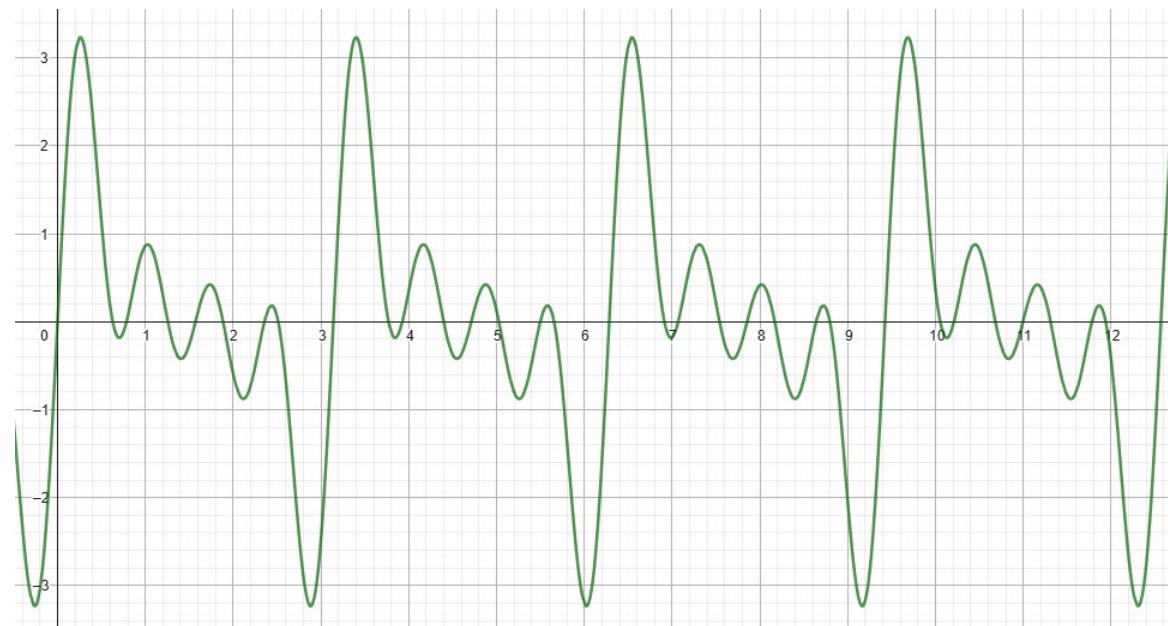
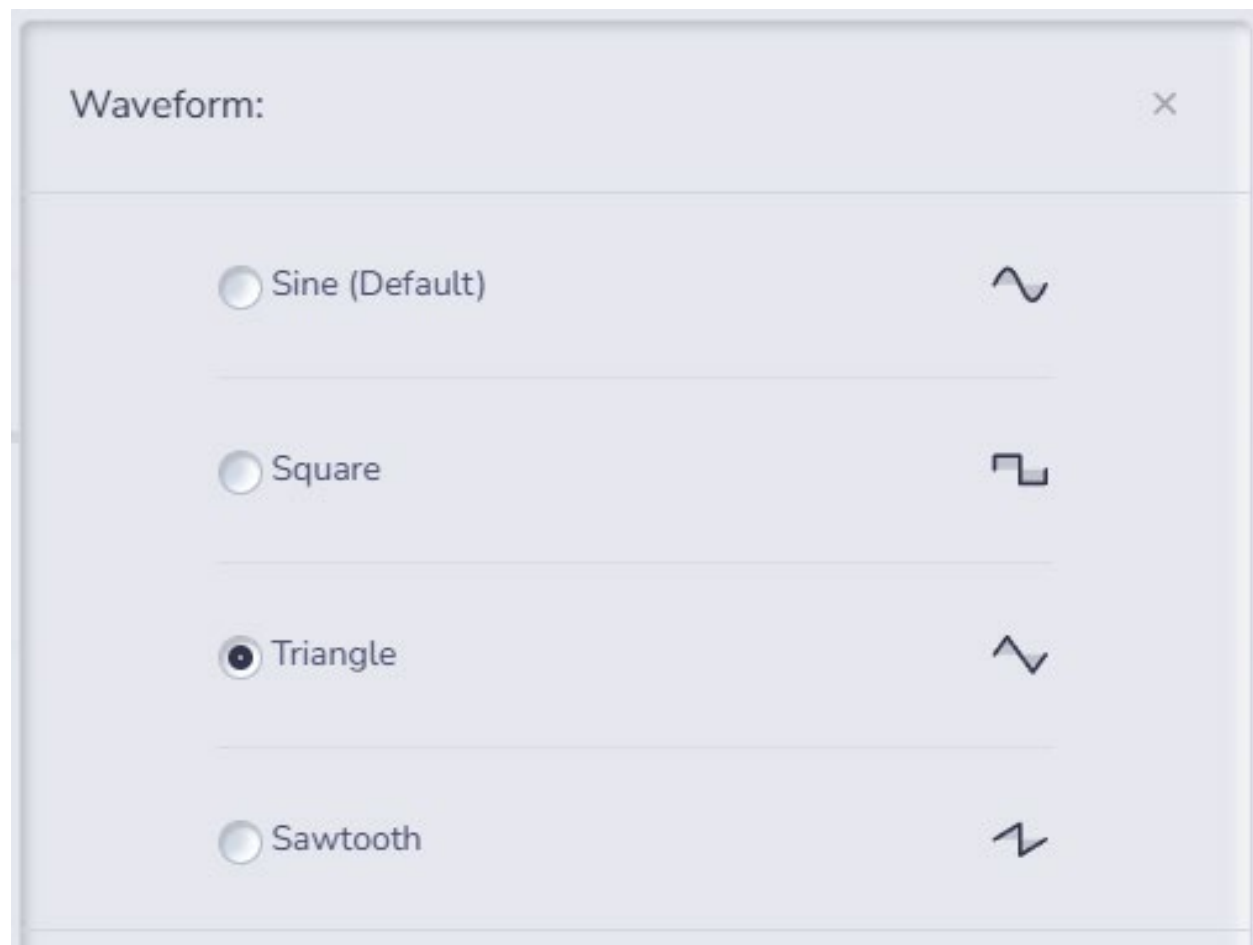
$$\begin{aligned} p(x) &= f(x) + g(x) + h(x) \\ &= \sin(x) + \sin(2x) + \sin(3x) \end{aligned}$$

Timbre – Square: odd coefficients



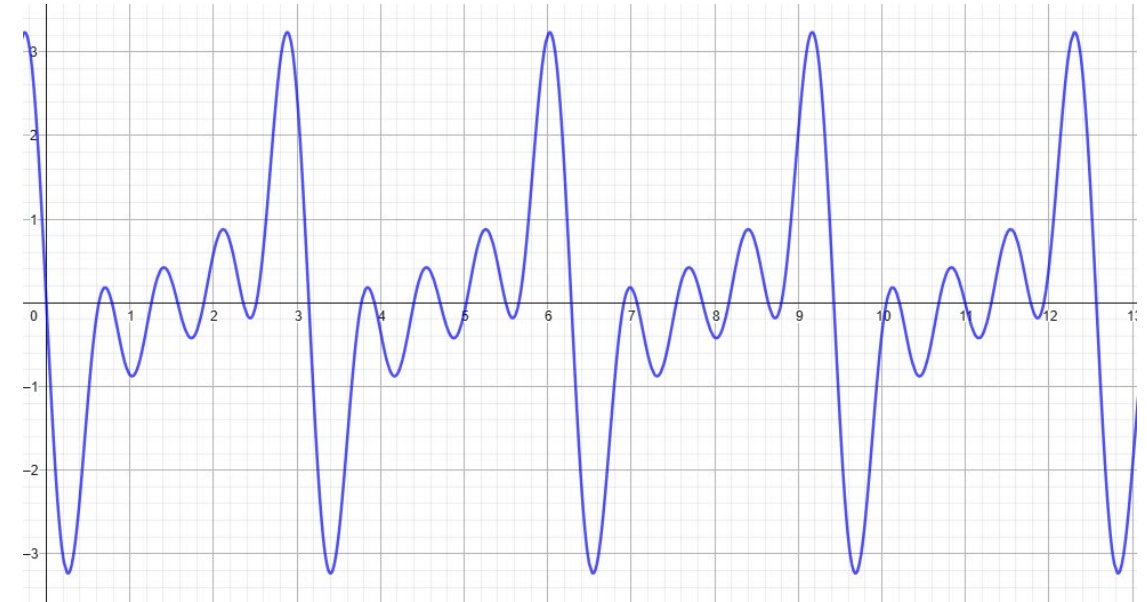
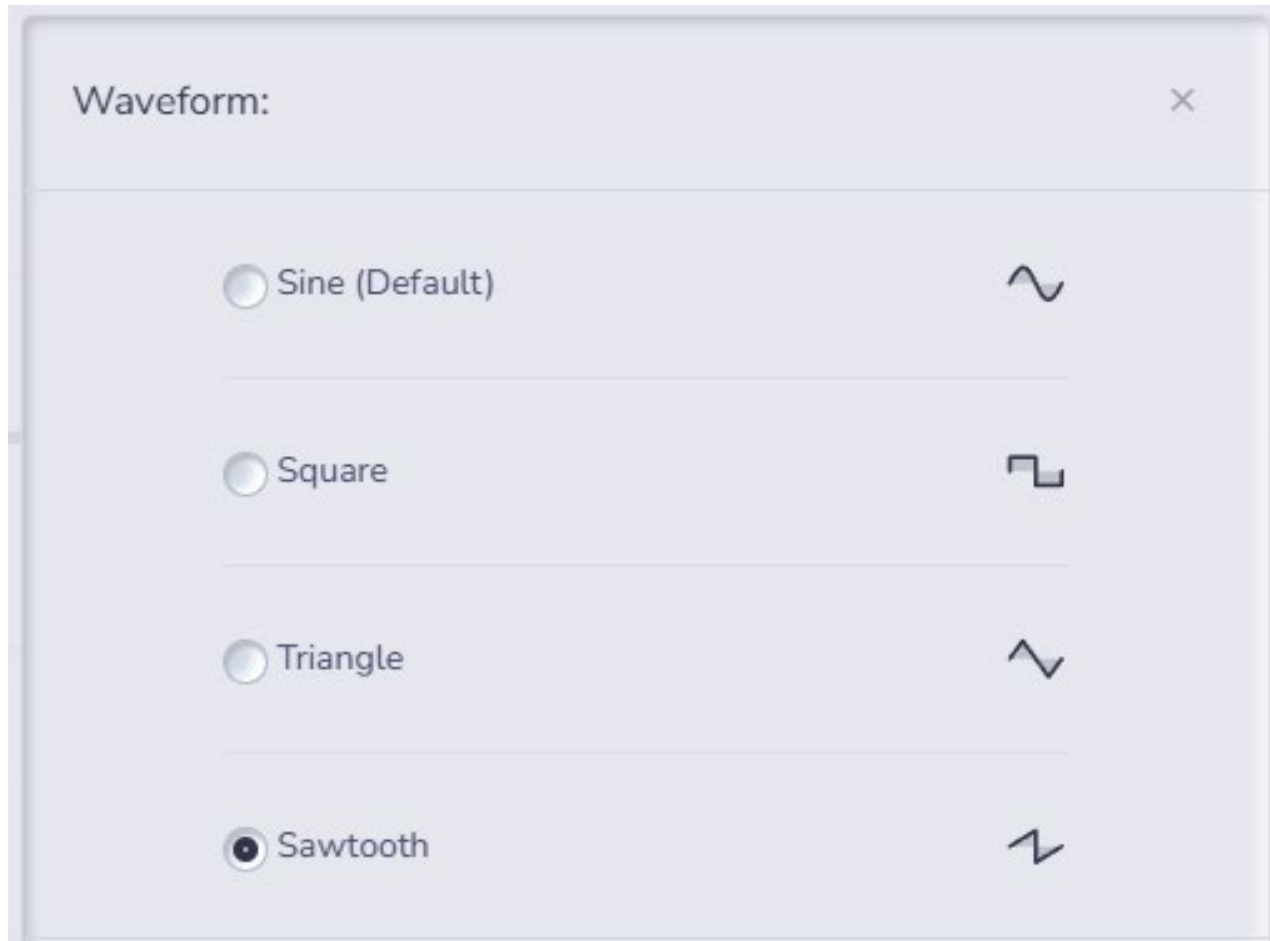
$$r: y = \sin(x) + \sin(3x) + \sin(5x) + \sin(7x)$$

Timbre – Triangle: even coefficients



$$q(x) = \sin(2x) + \sin(4x) + \sin(6x) + \sin(8x)$$

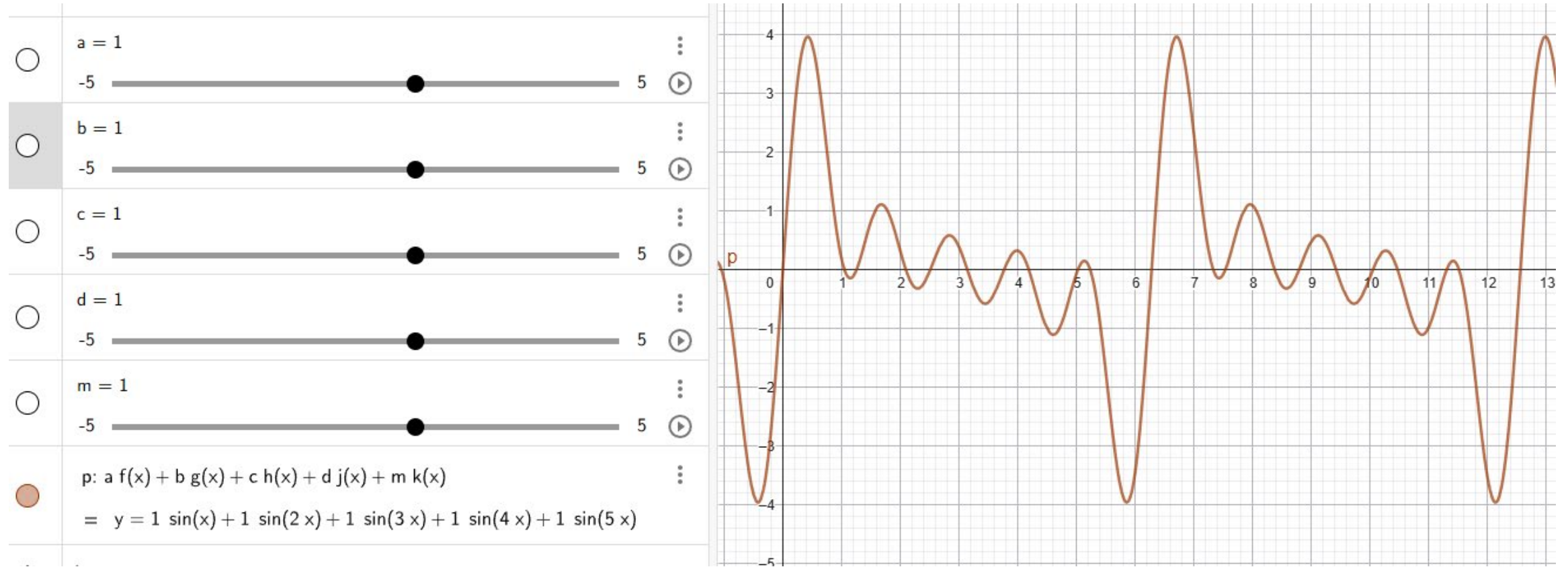
Timbre - Sawtooth



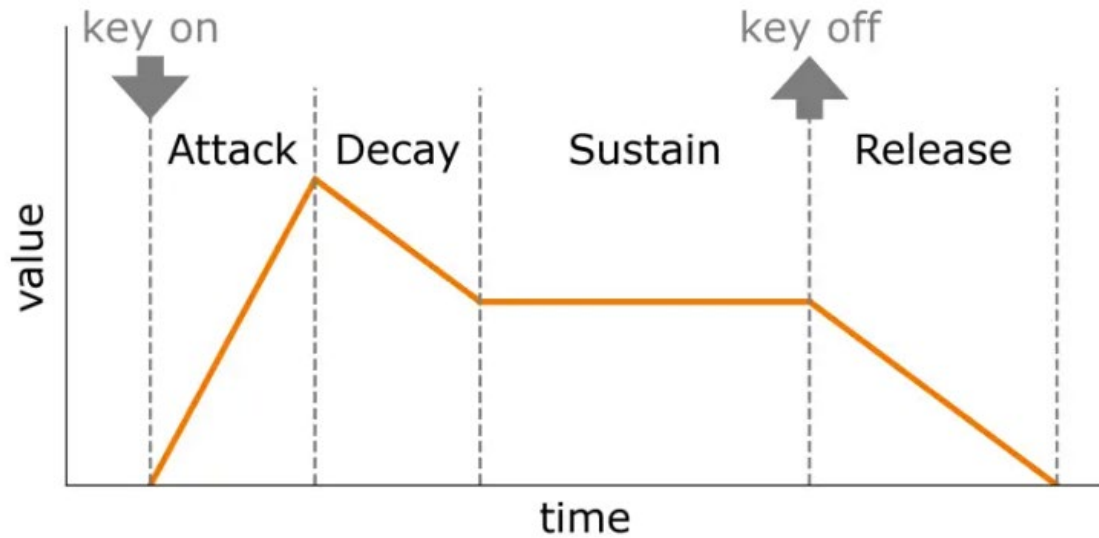
$$s(x) = -q(x)$$

$$= -(\sin(2x) + \sin(4x) + \sin(6x) + \sin(8x))$$

Harmonic Construction



ADSR Sound Envelope



- An ADSR envelope is a control mechanism in sound synthesis and music production that shapes how a sound changes over time.
- It stands for **Attack, Decay, Sustain, and Release**.
- It is most commonly used to control volume (amplitude)

What is the shape of the ADSR envelope for a single note played on:

1. Organ
2. Violin
3. Piano
4. Flute?

Rhythm

March 4/4

Waltz $\frac{3}{4}$

Triplets

3 against 2

Syncopation

Chopin B minor



'I got Rhythm'





I got
rhythm—
George
Gershwin

All that Jazz- Interesting intervals and chords

Outline

- Triads and inversions
- Ragtime
- Sevenths
- Ninths
- Flat fifths
- Sharp/ raised ninths
- Diatonic substitutions

All that Jazz- Interesting intervals and chords

Triads and inversions of major chords

- When the saints

Triads and inversions of minor chords

- Blue moon

Ragtime

- The Entertainer

CIRCLE OF FIFTHS

for Piano Club Singers

1

♩ = 60

Steven R

The musical score is written for three staves, each representing a different song. The time signature is 2/4, and the tempo is marked as ♩ = 60. The key signature has one flat (Bb). The chords are written in red text below each staff, following the Circle of Fifths progression: Dm, Gm, C, F, Bb, E dim, A, and Dm.

Staff	Song	Chord 1	Chord 2	Chord 3	Chord 4	Chord 5	Chord 6	Chord 7	Chord 8
1	Steven R	Dm	Gm	C	F	Bb	E dim	A	Dm
2	Fly me to Moon	Dm	Gm	C	F	Bb	E dim	A	Dm
3	Autumn Leaves	Dm	Gm	C	F	Bb	E dim	A	Dm

Fly me to Moon

Autumn Leaves

All that Jazz- more interesting intervals and chords

Sevenths

- Major 7th (CM7): Major triad + major 7th. Notes: (C - E - G - B). Has a dreamy, jazzy sound.
- Dominant 7th (C7): Major triad + minor 7th. Notes: (C - E - G - B $\flat$). Often creates a strong pull to resolve to the root note.
- Minor 7th (Cm7): Minor triad + minor 7th. Notes: (C - E $\flat$ - G - B $\flat$). A smooth, slightly melancholic sound common in R&B and pop.
- Minor-Major 7th (CmM7): Minor triad + major 7th. Notes: (C - E $\flat$ - G - B). Features a dramatic, suspenseful tension often heard in film scores.
- Diminished 7th (C $^\circ$ 7): Diminished triad + diminished 7th. Notes: (C - E $\flat$ - G $\flat$ - B $\flat$). Highly unstable, used for dramatic transitions.
- E.g. 'Misty'

All that Jazz- Interesting intervals and chords

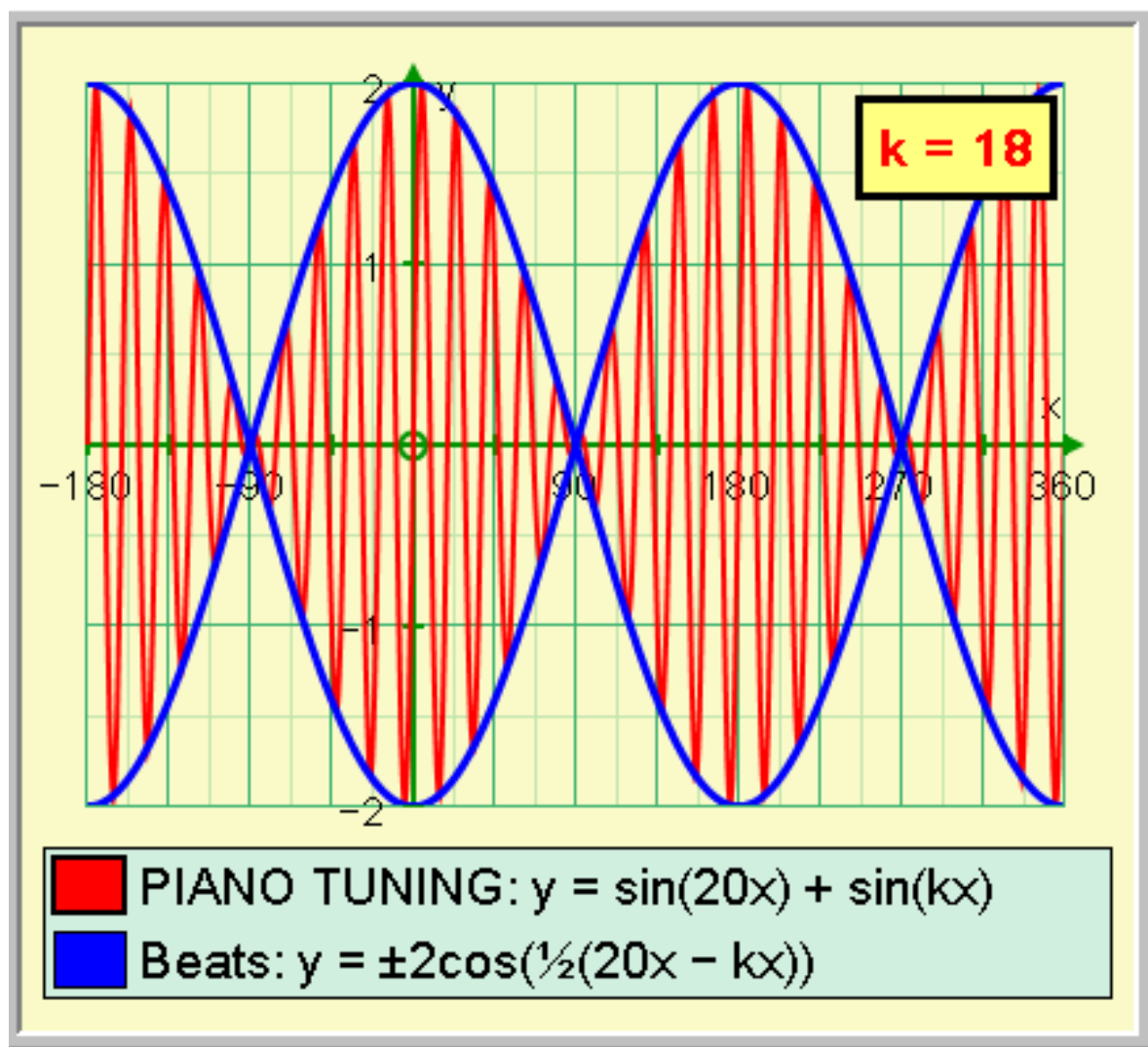
Ninths

- Moon River
- Always a woman to me – Billy Joel
- Flat fifths
- Sharp/ raised ninths
- Ipanema

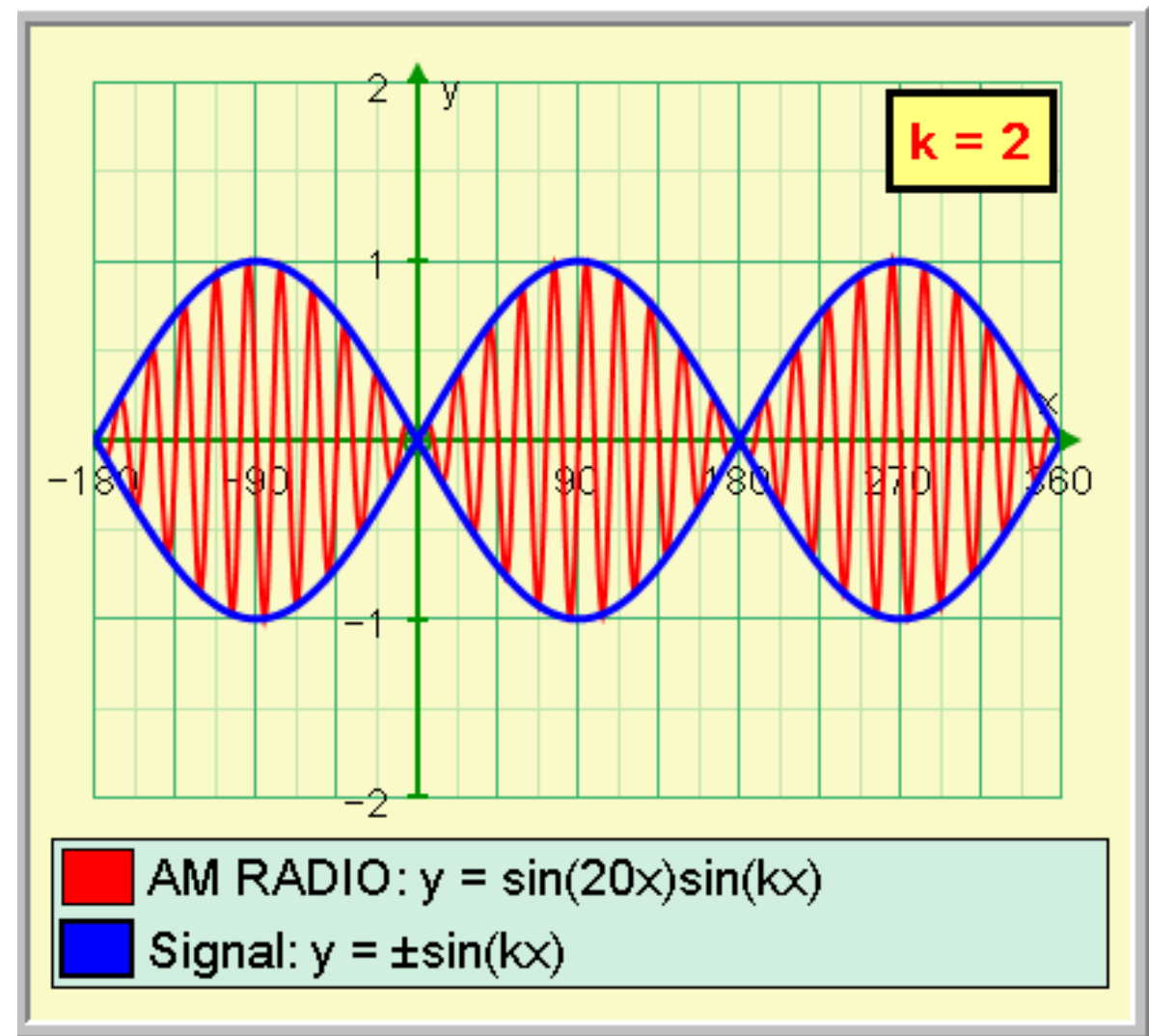
All that Jazz- Interesting intervals and chords

Diatonic substitutions – interchanging thirds and sevenths

- My foolish heart
- 6-2-5 turn around
-



$$\sin A + \sin B = 2\sin\frac{1}{2}(A + B) \cos\frac{1}{2}(A - B)$$



Trigonometry and Harmony

I got rhythm: Mathematics and Music

Douglas Butler - debutler17@gmail.com

Mick Blaylock – mickblaylock@live.com